New upper bounds on the chromatic number of a graph

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February 2, 2008

Abstract

We outline some ongoing work related to a conjecture of Reed [3] on ω, Δ, and χ. We conjecture that the complement of a counterexample G to Reed's conjecture has connectivity on the order of $\log(|G|)$. We prove that this holds for a family (parameterized by $\epsilon > 0$) of relaxed bounds; the $\epsilon = 0$ limit of which is Reed's upper bound.

1 Some Bounds

In all that follows, *graph* will mean finite simple graph with non-empty vertex set.
We will need the following result from [1].

Proposition 1. *Let $I_1, \ldots, I_m$ be disjoint independent sets in a graph G. Then*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + |G| - \sum_{j=1}^{m} |I_j| + 2m - 1\right). \tag{1}$$

In [2], an upper bound on the chromatic number in terms of the clique number, maximal degree and order was given.

Proposition 2. *Let G be a graph. Then,*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + \frac{|G| + \Delta(G) + 1}{2}\right).$$

We can do better than this when G has an induced subgraph with order much bigger than chromatic number.

Proposition 3. *Let G be a graph. Then, for any induced subgraph H of G,*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + \frac{\Delta(G) + 1 + |G|}{2}\right) + \frac{3\chi(H) - |H|}{4}.$$

Proposition 4. *Let G be a graph and K a cut-set in $\overline{G}$. Then, for any induced subgraph H of G,*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + \Delta(G) + 1\right) + \frac{4\chi(G[K]) + 3\chi(H \smallsetminus K) - |H \smallsetminus K|}{4}.$$

Corollary 5. *Let G be a graph. Then, for any induced subgraph H of G,*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + \Delta(G) + 1\right) + \frac{5\kappa(\overline{G}) + 3\chi(H) - |H|}{4}.$$

Corollary 6. *Let G be a graph and K a cut-set in $\overline{G}$. Then*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + \Delta(G) + 1\right) + \frac{4\chi(G[K]) + \alpha(G[K]) + 3 - \alpha(G)}{4}.$$

Corollary 7. *Let G be a graph. Then*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + \Delta(G) + 1\right) + \kappa(\overline{G}) + 1 - \frac{\alpha(G)}{4}.$$

2 Chromatic Excess

Definition 8. Let G be a graph. The *chromatic excess* of G is defined to be

$$\eta(G) = \max_{H \leq G} |H| - 3\chi(H).$$

The extremal cases of H being a maximal independent set and $H = G$ give the following constraints.

Lemma 9. $\alpha - 3 \leq \eta \leq \frac{\alpha-3}{\alpha}n$.

Lemma 10. $\eta \geq n - 3\chi$.

We may rewrite Proposition 3 and Corollary 5 in terms of the chromatic excess.

Corollary 11.

$$\chi \leq \frac{1}{2}\left(\omega + \frac{\Delta + 1 + n}{2}\right) - \frac{\eta}{4}.$$

Or equivalently.

Corollary 12.

$$\chi \leq \frac{1}{2}(\omega + \Delta + 1) + \frac{\overline{\delta} - \eta}{4}.$$

Corollary 13.

$$\chi \leq \frac{1}{2}\left(\omega + \Delta + 1\right) + \frac{5\overline{\kappa} - \eta}{4}.$$

3 A Counterexample To Reed's Conjecture Has Highly Connected Complement?

Proposition 14. *For every $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that for any graph G, at least one of the following holds*

- $\chi(G) \leq (\frac{1}{2} + \epsilon)\omega(G) + \frac{\Delta(G)+2}{2}$,
- $\kappa(\overline{G}) \geq C(\epsilon)\log(|G|)$.

Proof. Assume the former does not hold. Apply Corollary 7 to get upper bounds on α and ω in terms of $\overline{\kappa}$. Now use Ramsey Theory. $\square$

Conjecture 15. *There exists a constant $C > 0$ such that*

$$\chi > \left\lceil \frac{1}{2}(\omega + \Delta + 1) \right\rceil \Rightarrow \overline{\kappa} \geq C \log(n).$$

References

[1] landon rabern. On Graph Associations, *SIAM Journal on Discrete Mathematics*. Volume 20, Issue 2. Pages 529-535.

[2] landon rabern. A Note On Reed's Conjecture, arXiv:math.CO/0604499.

[3] Bruce Reed. ω, Δ, and χ, *J. Graph Theory* 27, (1997), 177–212.